PHYSICAL REVIEW B VOLUME 3, NUMBER 3 1 FEBRUARY 1971

Longitudinal Susceptibility of a Planar Ferromagnet

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(Received 13 July 1970)

The longitudinal susceptibility $\chi_{zz}(\vec{q}, \omega)$ of a spin-$\frac{1}{2}$ planar ferromagnet is investigated. The random-phase approximation is used and particular emphasis is placed on the long-wavelength behavior. The static susceptibility for temperatures in the vicinity of the Curie point is calculated. Correlation lengths above and below T_C are obtained. In the region below T_C the expression for $\chi_{zz}(\vec{q}, \omega)$ is compared with the susceptibility of an ideal magnon gas. Departures from ideal-gas behavior near the Curie point are studied. The relevance of the findings to recent hydrodynamic- and dynamic-scaling-law theories is discussed.

I. INTRODUCTION

The planar ferromagnet is one of the most interesting of the less intensively studied magnetic systems. The term planar refers to the fact that the anisotropy of the Hamiltonian is such that the system has an easy plane of magnetization perpendicular to some preferred axis. Recently, there has been theoretical research on the dynamical properties of this system in the critical region about the Curie point and in the hydrodynamic region below T_C.[1-3] In addition to being a magnetic system in its own right, the planar magnet has many features which make it suitable as a model for He^4 near and below the λ point. This was pointed out some time ago by Matsubara and Matsuda.[4] More recently, Whitlock and Zilsel have also emphasized the close connection between the two systems.[5] Unfortunately, there appears to be little experimental work on crystals which come close to having the properties of the ideal planar ferromagnet. If anisotropic, most magnetic systems have one or more easy axes of magnetization. It is hoped that the theoretical interest in the planar systems will stimulate further experimental efforts.

In this paper, the study of the susceptibilities of the spin-$\frac{1}{2}$ planar ferromagnet is continued.[6] The main interest is in the dynamic longitudinal susceptibility $\chi_{zz}(\vec{q}, \omega)$ (z is along the direction of magnetization) in the region near and below the Curie point. As a by-product, expressions for the Curie temperature and the spontaneous magnetization will be obtained. A Green's-function formalism first developed for isotropic ferromagnets by Liu[7] and later extended by Tanaka and Tani[8] to ferromagnets (and antiferromagnets) with an easy axis of magnetization is used. The essential approximation has come to be called the random-phase approximation (RPA). It is well known that the use of the RPA leads to a description of the magnetic system as an assembly of undamped quasiparticles whose energies are renormalized in a self-consistent manner. The absence of damping can be remedied in principle by a higher-order decoupling approximation. However, many features of real systems, particularly those pertaining to the static properties, are reproduced in a qualitative, if not semiquantitative, fashion by the RPA. It is felt for this reason that microscopic calculations of the RPA type are useful, although it is anticipated that they will be eventually superseded by more refined theories.

The remainder of the paper is divided into six parts. In Sec. II the essential features of the formal calculation are outlined. The static susceptibility, $\chi_{zz}(\vec{q}, 0)$ is studied in Sec. III, while in Sec. IV comment is made on the dynamic behavior. Section V consists of a discussion of the significance of the results, where correlation of the findings with the predictions of hydrodynamic- and dynamic-scaling-law theories is done. In Appendix A, the Curie temperature, the susceptibility above T_C, and the spontaneous magnetization are calculated while in Appendix B two complicated functions appearing in Sec. II are displayed.

II. FORMAL CALCULATION

This section outlines the calculation of the longitudinal susceptibility following the approach developed in Ref. 7. The Hamiltonian is written

$$\mathcal{H} = \mathcal{H}_0 + V, \tag{2.1}$$

where $\mathcal{H}_0$, the intrinsic Hamiltonian, has the form

$$\mathcal{H}_0 = -h\sum_i S_z^i - \sum_{i,j} J_{ij}\vec{S}^i \cdot \vec{S}^j - \sum_{i,j}(K_{ij} - J_{ij})S_x^i S_x^j \,. \tag{2.2}$$

The symbol h denotes the product $g\mu_B H$, with H being the external magnetic field. The symbols J_{ij} and K_{ij} denote exchange integrals, $\vec{S}$ is the spin ($|\vec{S}| = \frac{1}{2}$), and the sum is over the N spins in the lattice. A coordinate system is chosen such that the magnetization is in the z direction and x is perpendicular to the easy plane. The Hamiltonian V describes the coupling of the spin system to a space- and time-dependent external field. It is written

$$V = -f e^{-i\omega t + \epsilon t} \sum_i S_z^i e^{i\vec{q}\cdot\vec{r}_i} , \quad (2.3)$$

where $\epsilon \to 0+$.

In order to calculate the longitudinal susceptibility two Green's functions $G_{ij}^{+}(t)$ and $G_{ij}^{-}(t)$ are introduced which are defined by the equations

$$G_{ij}(t) = -i \langle T S_+^i(t) S_-^j(0) \rangle , \quad (2.4)$$

$$G_{ij}^{-}(t) = -i \langle T S_-^i(t) S_-^j(0) \rangle . \quad (2.5)$$

Here $S_\pm = S_x \pm i S_y$, and T denotes the time-ordering operator. As shown in detail in Ref. 7, the explicit time dependence of the Hamiltonian leads to a definition of the expectation values $\langle T S_\pm^i(t) S_-^j(0) \rangle$ in terms of the time-development and density operators at $t = -\infty$.

The RPA is introduced by replacing the operator $S_z^i(t)$ with its expectation value $\langle S_z^i(t) \rangle = r_i(t)$ in the equations of motion for $G^\pm$. The resulting equations are then analytically contined to the imaginary time (τ) domain. At this point, both the Green's function and r_i are expanded in powers of f:

$$G_{ij}^{\pm} = {}^0G_{ij}^{\pm} + f\,{}^1G_{ij}^{\pm} + O(f^2) , \quad (2.6)$$

$$r_i = \sigma + f\,{}^1r_i + O(f^2) , \quad (2.7)$$

where σ is the equilibrium value of S_z in the absence of the space- and time-dependent external field. To solve the zero-order equation, ${}^0G_{ij}^{\pm}$ is expanded in a Fourier series

$${}^0G_{ij}^{\pm}(\tau) = \frac{1}{N} \sum_{\vec{q}} \frac{1}{\beta} \sum_m {}^0G^{\pm}(\vec{q}, \omega_m)\, e^{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j) - i\omega_m \tau} , \quad (2.8)$$

where $\vec{q}$ runs over the Brillouin zone, $\beta = 1/kT$, and $\omega_m = 2\pi m/\beta$ with m taking on integer values from minus to plus infinity. The following solutions[6] are obtained:

$${}^0G^{+}(\vec{q}, \omega_m) = 2\sigma(A_{\vec{q}} + i\omega_m)/(\omega_m^2 + \omega_{\vec{q}}^2) , \quad (2.9)$$

$${}^0G^{-}(\vec{q}, \omega_m) = 2\sigma B_{\vec{q}}/(\omega_m^2 + \omega_{\vec{q}}^2) , \quad (2.10)$$

where

$$A_{\vec{q}} = \sigma[2J(0) - J(\vec{q}) - K(\vec{q})] + h , \quad (2.11)$$

$$B_{\vec{q}} = \sigma[K(\vec{q}) - J(\vec{q})] , \quad (2.12)$$

$$\omega_{\vec{q}}^2 = A_{\vec{q}}^2 - B_{\vec{q}}^2 , \quad (2.13)$$

with

$$J(\vec{q}) = \sum_j e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} J_{ij} , \quad (2.14)$$

$$K(\vec{q}) = \sum_j e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} K_{ij} . \quad (2.15)$$

Making use of the relationship

$$S_-^i S_+^i = \tfrac{1}{2} - S_z^i , \quad (2.16)$$

which is a special property of the spin-$\frac{1}{2}$ system, the self-consistent equations for r_i can be derived. In lowest order

$$\tfrac{1}{2} - \sigma = {}^0G_{ii}^{+}(-\delta) \quad (2.17)$$

in the limit $\delta \to 0+$. The implications of Eq. (2.17) are considered in detail in Appendix A.

The equations for the first-order Green's functions are far more complex. To solve them, the Fourier transforms are introduced,

$${}^1G_{ij}^{\pm}(\tau) = \frac{1}{N^2} \sum_{\vec{q}_1, \vec{q}_2} \frac{1}{\beta} \sum_m {}^1G^{\pm}(\vec{q}_1, \vec{q}_2, \omega_m)\, e^{i\vec{q}_1\cdot\vec{r}_i - i\vec{q}_2\cdot\vec{r}_j - i\omega_m\tau} , \quad (2.18)$$

$${}^1r_i(\tau) = \frac{1}{N} \sum_{\vec{q}} {}^1r(\vec{q}, \omega_n)\, e^{i\vec{q}\cdot\vec{r}_i - i\omega_n\tau} . \quad (2.19)$$

After a lengthy calculation, an equation for ${}^1G^{+}$ which involves ${}^1r(\vec{q}_1 - \vec{q}_2, \omega_n)$ is obtained. It is possible to solve for the latter function by making use of the first-order equation[7]

$${}^1r(\vec{q}, \omega_n) = -\frac{1}{N} \sum_{q_1} \frac{1}{\beta} \sum_m {}^1G^{+}(\vec{q}_1, \vec{q}_1 - \vec{q}, \omega_m)\, e^{i\delta\omega_m} . \quad (2.20)$$

As a result

$${}^1r(\vec{q}, \omega_n) = \sum_{\vec{q}_1} U_1(\vec{q}_1, \vec{q}_1 - \vec{q}, \omega_n) \Big(1 + \frac{1}{N} \sum_{\vec{q}_1} \{ [K(\vec{q}_1 - \vec{q}) + J(\vec{q}_1 - \vec{q}) - 2J(\vec{q})]\, U_1(\vec{q}_1, \vec{q}_1 - \vec{q}, \omega_n) + [K(\vec{q}_1 - \vec{q}) - J(\vec{q}_1 - \vec{q})]\, U_2(\vec{q}_1, \vec{q}_1 - \vec{q}, \omega_n) \} \Big)^{-1} , \quad (2.21)$$

where the functions U_1 and U_2 are given in closed form by Eqs. (B1) and (B2).

As is apparent from the discussion in Sec. II of Ref. 7, the dynamic susceptibility $\chi_{zz}(\vec{q}, \omega)$ is obtained by the analytic continuation of ${}^1r(\vec{q}, \omega_n)$ into the complex frequency plane:

$$\chi_{zz}(\vec{q}, \omega) = {}^1r(\vec{q}, i\omega - \delta) . \quad (2.22)$$

Equations (2.21) and (2.22) are the principal results of this section.

III. STATIC SUSCEPTIBILITY

In this section, the static susceptibility $\chi_{zz}(\vec{q}, 0)$ is examined. Recently, some controversy has arisen over the connection between the static limit of $\chi_{zz}(\vec{q}, \omega)$, which is sometimes referred to as the isolated susceptibility, and the thermodynamic adiabatic and isothermal susceptibilities. Falk[9] and Wilcox[10] have established that the isothermal susceptibility is bounded from above by the adiabatic susceptibility, which in turn is bounded from above by the isothermal susceptibility. Although model systems have been found for which the isolated susceptibility differs from the adiabatic, it has been argued that for realistic systems they are identical in the thermodynamic limit.[10] (In the present analysis, the distinction between isothermal and adiabatic disappears for finite q since $\sum_i e^{i\vec{q}\cdot\vec{r}_i}\langle S_z^i\rangle = 0$ for $\vec{q}\neq 0$.) In light of this, interpreting the expression for $\chi_{zz}(\vec{q}, 0)$ as an approximation to the susceptibility which would be measured in a hypothetical experiment with a time-independent external field is felt to be justified.

At this point, the analysis is limited to wave vectors $\vec{q}$ such that the corresponding energies $\omega_{\vec{q}}$ are much less than kT. It is found that

$$U_1(\vec{q}_1, \vec{q}_2, 0) = 4\sigma^2(A_{\vec{q}_1}A_{\vec{q}_2} + B_{\vec{q}_1}B_{\vec{q}_2})/\beta\omega_{\vec{q}_1}^2\omega_{\vec{q}_2}^2 , \tag{3.1}$$

$$U_2(\vec{q}_1, \vec{q}_2, 0) = 4\sigma^2(A_{\vec{q}_1}B_{\vec{q}_2} + B_{\vec{q}_1}A_{\vec{q}_2})/\beta\omega_{\vec{q}_1}^2\omega_{\vec{q}_2}^2 . \tag{3.2}$$

These two equations lead to the result

$$\chi_{zz}(\vec{q}, 0) = \frac{4\sigma^2}{\beta}\sum_{\vec{q}_1}\frac{A_{\vec{q}_1}A_{\vec{q}_1-\vec{q}} + B_{\vec{q}_1}B_{\vec{q}_1-\vec{q}}}{\omega_{\vec{q}_1}^2\,\omega_{\vec{q}_1-\vec{q}}^2}\left(\frac{4\sigma}{\beta}\left[2\sigma(J(0)-J(\vec{q}))+h\right]\frac{1}{N}\sum_{\vec{q}_1}\frac{A_{\vec{q}_1}A_{\vec{q}_1-\vec{q}} + B_{\vec{q}_1}B_{\vec{q}_1-\vec{q}}}{\omega_{\vec{q}_1}^2\,\omega_{\vec{q}_1-\vec{q}}^2} + 1 - \frac{4\sigma}{\beta N}\sum_{\vec{q}_1}\frac{A_{\vec{q}_1}}{\omega_{\vec{q}_1}^2}\right)^{-1} . \tag{3.3}$$

Above T_C in the limit $h\to 0$, $h/\sigma = 1/\chi_\perp$, where $\chi_\perp$ is the uniform-field susceptibility in the easy plane. With the help of Eq. (A1), the following equation is obtained:

$$\chi_{zz}(\vec{q}, 0) = \frac{N\chi_\perp}{1 + 2\chi_\perp[J(0) - J(\vec{q})]} , \tag{3.4}$$

which is identical to the equation for $\chi_{yy}(\vec{q}, 0)$ derived in Ref. 1. An approximate expression for $\chi_\perp$ which is appropriate near the Curie point is given in Eq. (A4).

Below the ordering temperature we make use of Eq. (A2) which leads to

$$\chi_{zz}(\vec{q}, 0) = \frac{\chi_{zz}(\vec{q}, 0)_I}{2[J(0) - J(\vec{q})]N^{-1}\chi_{zz}(\vec{q}, 0)_I + (1 - \beta_C/\beta)} , \tag{3.5}$$

where $\beta_C = 1/kT_C$ and

$$\chi_{zz}(\vec{q}, 0)_I = \frac{4\sigma^2}{\beta}\sum_{\vec{q}_1}\frac{(A_{\vec{q}_1}A_{\vec{q}_1-\vec{q}} + B_{\vec{q}_1}B_{\vec{q}_1-\vec{q}})}{\omega_{\vec{q}_1}^2\,\omega_{\vec{q}_1-\vec{q}}^2} , \tag{3.6}$$

is the static susceptibility of an ideal magnon gas.

In the long-wavelength limit it is possible to write

$$A_0 = -B_0 , \tag{3.7}$$

$$\omega_{\vec{q}} = 2\sigma c q , \tag{3.8}$$

where

$$c = D^{1/2}[J(0) - K(0)]^{1/2} a , \tag{3.9}$$

with D being defined by

$$J(0) - J(\vec{q}) = Da^2q^2 + O(q^4) , \tag{3.10}$$

with $a^3 = V/N$, V being the volume of the system. As a result[11]

$$\chi_{zz}(\vec{q}, 0)_I = \frac{A_0^2 V}{2\beta c^4\sigma^2}\,\frac{1}{(2\pi)^3}\int\frac{d\vec{q}}{\vec{q}_1^2(\vec{q}_1 - \vec{q})^2} = \frac{A_0^2 V}{32\beta c^4\sigma^2 q} . \tag{3.11}$$

Thus, below T_C the susceptibility can be written

$$\chi_{zz}(\vec{q}, 0) = \frac{N}{32D^2qa(\beta - \beta_C)}\left(1 + \frac{qa}{16D(\beta - \beta_C)}\right)^{-1} . \tag{3.12}$$

Up to this point the calculations have been carried out for finite temperatures with the restriction $\omega_{\vec{q}} \ll kT$. It is apparent that in this limit, $\chi_{zz}(\vec{q}, 0)$ diverges as $1/q$ when q approaches zero. Quite different behavior is obtained if the susceptibility is evaluated at absolute zero. At $T = 0$, $\chi_{zz}(\vec{q}, 0)_I$ can be written

$$\chi_{zz}(\vec{q}, 0)_I = A_0^2\sum_{\vec{q}_1}\frac{1}{\omega_{\vec{q}_1}\omega_{\vec{q}_1-\vec{q}}(\omega_{\vec{q}_1} + \omega_{\vec{q}_1-\vec{q}})} . \tag{3.13}$$

The evaluation of the integral in (3.13) is straightforward with the result

$$\chi_{zz}(\vec{q}, 0)_I = \frac{A_0^{1/2}N(-\ln q)}{8\sqrt{2}\pi^2 D^{3/2}} \tag{3.14}$$

in the limit $q\to 0$.

The finite value of $\chi_{zz}(\vec{q}, 0)_I$ reflects the fact that the spins have a small zero-point motion, and hence are susceptible to enhanced alignment by an external field. Such is not the case for isotropic ferromagnets and ferromagnets with an easy axis of magnetization which are in a state of maximum spin alignment at $T = 0$. For these systems, the susceptibility is zero at $T = 0$.

IV. DYNAMIC SUSCEPTIBILITY

In this section the long-wavelength behavior of the dynamic susceptibility in the region below T_C is examined. Consider first, the numerator of

Eq. (2.21). As is to be anticipated from the discussion in Sec. III, the numerator, when analytically continued, yields the dynamic susceptibility of the ideal magnon gas $\chi_{zz}(\vec{q}, \omega)_I$. Thus,

$$\chi_{zz}(\vec{q}, \omega)_I = \sum_{\vec{q}_1} U_1(\vec{q}_1, \vec{q}_1 - \vec{q}, i\omega - \delta) . \quad (4.1)$$

At long wavelengths, where $\omega_{\vec{q}} = 2\sigma c q$, the imaginary part of the susceptibility in closed form $(h=0)$ can be written

$$\chi_{zz}''(\vec{q}, \omega)_I = \frac{N}{16\pi D^2 q a \beta} \ln\left(\frac{\sinh\frac{1}{4}\beta|\omega + \omega_q|}{\sinh\frac{1}{2}\beta|\omega - \omega_q|}\right) . \quad (4.2)$$

At finite temperatures, $\chi_{zz}''(q, \omega)_I$ has a logarithmic singularity at $\omega = \pm\omega_{\vec{q}}$. In the zero-temperature limit, the following result is obtained:

$$\chi_{zz}''(\vec{q}, \omega) = \frac{N}{64\pi D^2 q a} (|\omega + \omega_{\vec{q}}| - |\omega - \omega_{\vec{q}}|) ,$$

which is the counterpart of Eq. (3.14). The weak singularity is seen to disappear at $T=0$.

As T approaches T_C, corrections to (4.2) become important for all but the longest wavelengths. Something of the nature of these corrections can be inferred by an approximate evaluation of the denominator of the right-hand side of (2.21). After replacing ω_n by $i\omega - \delta$, the denominator of (2.21) becomes $(q \to 0)$

$$1 - \frac{\beta_C}{\beta} - \frac{i\sigma^2 D A_0^2 \omega a^5}{\pi(2\sigma c)^5 \beta} , \quad (4.3)$$

after having approximated the real part by its value at $\omega = 0$.

Combining (4.3) and (4.1), an approximate expression for $\chi_{zz}(\vec{q}, \omega)$ is obtained,

$$\chi_{zz}(\vec{q}, \omega) = \frac{\chi_{zz}(\vec{q}, \omega)_I}{(1 - \beta_C/\beta)(1 - i\omega\tau)} , \quad (4.4)$$

where τ is given by

$$\tau = \frac{D A_0^2 a^5}{32\pi\sigma^3 c^5 (\beta - \beta_C)} . \quad (4.5)$$

As is apparent from (4.2), the susceptibility of the ideal magnon gas depends on frequency only in the ratio $\omega/\omega_{\vec{q}}$. Thus, if $\omega_{\vec{q}}\tau \ll 1$,

$$\chi_{zz}(\vec{q}, \omega) = \frac{\chi_{zz}(\vec{q}, \omega)_I}{1 - \beta_C/\beta} . \quad (4.6)$$

At low temperatures, τ approaches zero so that the condition $\omega_{\vec{q}}\tau \ll 1$ is satisfied over the entire Brillouin zone. Near the Curie point, τ becomes very large and the condition $\omega_{\vec{q}}\tau \ll 1$ can only be satisfied in a vanishing region about $\vec{q} = 0$. The significance of this result will be discussed in Sec. V.

V. DISCUSSION

The most interesting aspects of the calculation pertain to the behavior near the Curie point. The findings for the static susceptibility can be summarized as follows:

$$\chi_{zz}(\vec{q}, 0) \propto \frac{\xi_>^2}{1 + q^2\xi_>^2} , \quad T > T_C \quad (5.1)$$

$$\chi_{zz}(\vec{q}, 0) \propto \frac{\xi_<^{\parallel}}{q(1 + q\xi_<^{\parallel})} , \quad T < T_C \quad (5.2)$$

where $\xi_>$, the correlation length for fluctuations in S_z above T_C, is given by

$$\xi_> = (2D\chi_\perp)^{1/2} a = \frac{a}{4D\pi(\beta_C - \beta)} , \quad (5.3)$$

with $a = (V/N)^{1/3}$, i.e., the cube root of the volume per spin. Also, $\xi_<^{\parallel}$, the correlation length for fluctuations in S_z below T_C, is given by

$$\xi_<^{\parallel} = \frac{D A_0^2 a^5}{16 c^4 \sigma^2 (\beta - \beta_C)} = \frac{a}{16 D(\beta - \beta_C)} . \quad (5.4)$$

Below T_C in addition to $\xi_<^{\parallel}$, one can define a second correlation length $\xi_<^{\perp}$ which characterizes the fluctuations in S_y.[12] In the notation of this paper it is written

$$\frac{\sigma^2 \xi_<^{\perp}}{r} = \frac{1}{N} \sum_{\vec{q}} \frac{\chi_{yy}(\vec{q}, 0) e^{i\vec{q}\cdot\vec{r}}}{\beta_C} . \quad (5.5)$$

In the RPA, $\chi_{yy}(\vec{q}, 0)$ can be written[6]

$$\chi_{yy}(\vec{q}, 0) = \frac{N}{2[J(0) - J(\vec{q})]} = \frac{N}{2Dq^2a^2} \quad (5.6)$$

for small q. From (5.5) and (5.6), the result is

$$\xi_<^{\perp} = a/8\pi D\beta_C\sigma^2 , \quad (5.7)$$

which in light of (A4) reduces to

$$\xi_<^{\perp} = \beta_C J(0) a / 12\pi(\beta - \beta_C) D . \quad (5.8)$$

An interesting feature of the calculation relates to the ratios of the correlation lengths for temperatures ΔT above and below T_C,

$$\xi_<^{\parallel}(T_C - \Delta T) = \tfrac{1}{4}\pi\xi_>(T_C + \Delta T) , \quad (5.9)$$

$$\xi_<^{\perp}(T_C - \Delta T) = \frac{4\beta_C J(0)}{3\pi} \xi_<^{\parallel}(T_C - \Delta T) . \quad (5.10)$$

As for the dynamic longitudinal susceptibility, the results indicate that $\beta \to \beta_C$, the ideal (but renormalized) spin-wave behavior is restricted to a vanishing region about the center of the zone where the condition $\omega_{\vec{q}}\tau \ll 1$ can be satisfied. Making use of (3.8) and (5.4), τ can be rewritten as

$$\tau = \xi_<^{\parallel} q / \pi\omega_{\vec{q}} . \quad (5.11)$$

Equations (4.4), (4.6), and (5.11) show that there is a significant difference in the dynamics of the longitudinal fluctuations between the two regions $q\xi_<^{\parallel} \ll 1$ and $q\xi_<^{\parallel} \gtrsim 1$. This difference mirrors a corresponding difference in the static correlation.

The behavior of $\chi_{zz}(\vec{q}, \omega)$ for $T < T_C$ is similar to what is postulated in the dynamic-scaling-law hypothesis.[12] According to scaling-law theory, below the ordering temperature the spin-wave picture is valid as long as $q\xi_<^{\parallel} \ll 1$. For $q\xi_<^{\parallel} \gtrsim 1$ the dynamical behavior is more complex. However, the characteristic frequency ω_C, which in the present analysis may be defined by the integral

$$\int_{-\omega_C}^{\omega_C} \omega^{-1}\chi''_{zz}(\vec{q}, \omega)\, d\omega = \tfrac{1}{2}\int_{-\infty}^{\infty} \omega^{-1}\chi''_{zz}(\vec{q}, \omega)\, d\omega , \qquad (5.12)$$

is predicted to be of the form $q^z\Omega(q\xi_<^{\parallel})$. Since $\omega_{\vec{q}}$ is proportional to $q^{3/2}(q\xi_<^{\parallel})^{-1/2}$ and τ to $\xi_<^{\parallel 3/2}$ it is apparent that the characteristic frequency obtained with the expression for $\chi''_{zz}(\vec{q},\omega)$, (4.4), is of the form predicted by the dynamic-scaling hypothesis with $z=\frac{3}{2}$ and $\Omega(x) \propto x^{-1/2}$ as $x \to 0$. Furthermore, the $q^{3/2}$ behavior is in agreement with the predictions of Ref. 2.

It should be noted that the RPA can only reflect scaling-law behavior at finite temperatures below T_C since the quasiparticle energies vanish for $T \geq T_C$ $(h=0)$. Consequently, the theory has no dynamical content above the ordering temperature.

In the RPA, the longitudinal susceptibility of a planar ferromagnet is quite similar to the longitudinal susceptibility of an isotropic ferromagnet and the longitudinal staggered susceptibility of an isotropic antiferromagnet. In all three cases, a behavior is obtained of the form[8,11]

$$\chi_{zz}(\vec{q}, 0) \propto \xi_>^2/(1+q^2\xi_>^2) \qquad (5.13)$$

for the critical region above T_C, while below T_C they all vary according to

$$\chi_{zz}(\vec{q}, 0) \propto \xi_<^{\parallel}/q(1+q\xi_<^{\parallel}). \qquad (5.14)$$

The magnitudes of the correlation lengths are not the same for the three systems. However, in each case we obtain the relationship

$$\xi_<^{\parallel}(T_C-\Delta T) = \tfrac{1}{4}\pi\xi_>(T_C+\Delta T) . \qquad (5.15)$$

The susceptibility of the ferromagnet with an easy axis of magnetization has the Ornstein-Zernike form both above and below T_C with the correlation lengths being related by[8]

$$\xi_<^{\parallel}(T_C-\Delta T) = (1/\sqrt{2})\, \xi_>(T_C+\Delta T) .$$

The peculiar behavior of the susceptibility of the easy-axis ferromagnet, which has the form predicted by molecular field theory, can be traced to the presence of an anisotropy gap in the spin-wave spectrum at $q=0$, which is not found in the isotropic and planar systems.

The dynamic susceptibility below T_C also has many features in common with the dynamic (staggered) susceptibility of the isotropic (anti-) ferromagnet. For sufficiently small wave vectors, $\chi_{zz}(\vec{q}, \omega)$ is approximately equal to $\chi_{zz}(\vec{q}, \omega)_I$, the ideal spin-wave susceptibility. In each case, $\chi''_{zz}(\vec{q}, \omega)_I$ has a logarithmic singularity at $\omega = \pm\omega_{\vec{q}}$, where $\omega_{\vec{q}}$ is the corresponding spin-wave frequency.[13,14]

Although the excitations of the system have an infinite lifetime in this calculation, the relationship between frequency and susceptibility is the same as found in the hydrodynamic theory of spin waves.[3] According to the theory developed in Ref. 3, the spin-wave theory in the hydrodynamic (collision-dominated) region can be written in terms of the static susceptibilities

$$\omega_{\vec{q}} = \frac{N\sigma}{[\chi_{xx}(\vec{q}, 0)\,\chi_{yy}(\vec{q}, 0)]^{1/2}} . \qquad (5.16)$$

In the RPA, χ_{yy} is given by (5.6) and χ_{xx} by[6]

$$\chi_{xx}(\vec{q}, 0) = \frac{N}{2[J(0)-K(\vec{q})]} . \qquad (5.17)$$

With the help of (5.6) and (5.17), Eq. (5.16) can be written

$$\omega_{\vec{q}} = 2\sigma[J(0)-J(\vec{q})]^{1/2}\,[J(0)-K(\vec{q})]^{1/2} , \qquad (5.18)$$

which is the same as the frequency given by Eq. (2.13). This result is not surprising since in the RPA the static susceptibilities are the zero-frequency limits of dynamic susceptibilities which involve only three parameters σ, $A_{\vec{q}}$, and $B_{\vec{q}}$, the last two being directly related to frequency as shown in (2.13).

On the other hand, the behavior of $\chi_{zz}(\vec{q}, \omega)$ in the hydrodynamic theory is not the same as is found in the present calculation which is appropriate only to the "collisionless" regime. In Ref. 3, $\chi_{zz}(\vec{q}, \omega)$ is written as the sum of two terms, one of which is diffusive in character and reflects the coupling of the magnetization and energy-density fluctuations. The other term is inversely proportional to q, but has a frequency dependence characteristic of the time it takes to reach a state of local thermal equilibrium.

Finally, we would like to comment briefly on the connection between the planar ferromagnet and various models of He^4.[4,5,15,16] In the theories developed in Refs. 4, 5, and 15, models are formulated which are formally identical to a spin-$\frac{1}{2}$ planar ferromagnet in an external field *perpendicular* to the easy plane. In particular, the Hamiltonian adopted by Whitlock and Zilsel[5] can be written in our notation in the form

$$\mathcal{H} = -2J\sum_{\langle ij\rangle} \vec{S}^i \cdot \vec{S}^j - 2(J-K)\sum_{\langle ij\rangle} S_x^i S_x^j - 2Z(J-K)\sum_i S_x^i ,$$

where $\langle ij\rangle$ stands for nearest-neighbor pairs, and Z is the number of nearest neighbors. Further aspects of the quantal lattice-gas analogy are reviewed by Fisher.[16] In a recent paper on critical dynamics, Kawasaki has outlined a theory applicable near the λ point which is closer in spirit to the present calculation.[17] In his model, S_x, S_y, and S_z are identified with the entropy density and the two components of the order parameter. He derives approximate equations of motion for the fluctuations in these variables which are equivalent to the equations of motion of the spin operators in the *absence* of an external field.

APPENDIX A

In this Appendix, some of the results which follow from Eq. (2.17) are displayed. This equation can be written[6]

$$\frac{1}{2\sigma}=\frac{1}{N}\sum_{\vec{q}}\frac{A_{\vec{q}}}{\omega_{\vec{q}}}\coth\tfrac{1}{2}\beta\omega_{\vec{q}}\ . \qquad \text{(A1)}$$

By the usual arguments,[6] it is possible to derive an expression for the inverse Curie temperature of the form

$$\beta_C=\frac{4\sigma}{N}\sum_{\vec{q}}\frac{A_q}{\omega_q^2}\ . \qquad \text{(A2)}$$

In the case of a fcc lattice with nearest-neighbor interactions, use can be made of the theory of Flax and Raich[18] to evaluate the integrals in (A2),

$$\beta_C=2.69/J(0), \quad K(0)/J(0)=1\ ,$$

$$\beta_C=2.37/J(0), \quad K(0)/J(0)=\tfrac{1}{2}\ ,$$

$$\beta_C=2.35/J(0), \quad K(0)/J(0)=0\ .$$

The values for $K(0)/J(0)=1$, 0 are to be compared with the values obtained from series expansions (Refs. 19 and 20, respectively)

$$\beta_C=2.99/J(0), \quad K(0)/J(0)=1,$$

$$\beta_C=2.65/J(0), \quad K(0)/J(0)=0.$$

It is apparent that the anisotropic terms in the Hamiltonian raise the ordering temperature. This can be attributed to a limiting of the spin fluctuations perpendicular to the easy plane, as reflected in the fact that $\chi_{xx}(0,0)$ remains finite at T_C.

The magnetization near the Curie point takes the form

$$\sigma^2=[3/2\beta_C J(0)](1-\beta_C/\beta). \qquad \text{(A3)}$$

Immediately above T_C, the uniform-field susceptibility is given by

$$\chi_\perp=\frac{1}{32D^3\beta_C^2\pi^2(\beta_C/\beta-1)^2}\ . \qquad \text{(A4)}$$

The behavior shown in Eqs. (A3) and (A4) is similar to what is found for the isotropic ferromagnet in the RPA.[7,11] However, the divergence in the susceptibility is more severe than is inferred from series expansions. In particular, when $K(0)/J(0)=0$, Betts *et al.* find[20]

$$\chi_\perp\propto(T-T_C)^{-1.35}\ .$$

APPENDIX B

We have

$$\frac{U_1(\vec{q}_1,\vec{q}_2,\omega_n)}{4\sigma^2}=\frac{N_{\vec{q}_1}+1}{2\omega_{\vec{q}_1}}\left(\frac{(A_{\vec{q}_1}-\omega_{\vec{q}_1})(A_{\vec{q}_2}-\omega_{\vec{q}_1}-i\omega_n)+B_{\vec{q}_1}B_{\vec{q}_2}}{\omega_{\vec{q}_2}^2-(\omega_{\vec{q}_1}+i\omega_n)^2}\right)+\frac{N_{\vec{q}_2}+1}{2\omega_{\vec{q}_2}}\left(\frac{(A_{\vec{q}_1}-\omega_{\vec{q}_2}+i\omega_n)(A_{\vec{q}_2}-\omega_{\vec{q}_2})+B_{\vec{q}_1}B_{\vec{q}_2}}{\omega_{\vec{q}_1}^2-(\omega_{\vec{q}_2}-i\omega_n)^2}\right)$$

$$+\frac{N_{\vec{q}_1}}{2\omega_{\vec{q}_1}}\left(\frac{(A_{\vec{q}_1}+\omega_{\vec{q}_1})(A_{\vec{q}_2}+\omega_{\vec{q}_1}-i\omega_n)+B_{\vec{q}_1}B_{\vec{q}_2}}{\omega_{\vec{q}_2}^2-(\omega_{\vec{q}_1}-i\omega_n)^2}\right)+\frac{N_{\vec{q}_2}}{2\omega_{\vec{q}_2}}\left(\frac{(A_{\vec{q}_1}+\omega_{\vec{q}_2}+i\omega_n)(A_{\vec{q}_2}+\omega_{\vec{q}_2})+B_{\vec{q}_1}B_{\vec{q}_2}}{\omega_{\vec{q}_1}^2-(\omega_{\vec{q}_2}+i\omega_n)^2}\right)\ ,$$

where $N_{\vec{q}}=(e^{\beta\omega_{\vec{q}}}-1)^{-1}$, and

$$\frac{U_2(\vec{q}_1,\vec{q}_2,\omega_n)}{4\sigma^2}=\frac{N_{\vec{q}_1}+1}{2\omega_{\vec{q}_1}}\left(\frac{(A_{\vec{q}_1}-\omega_{\vec{q}_1})B_{\vec{q}_2}+(A_{\vec{q}_2}-\omega_{\vec{q}_1}-i\omega_n)B_{\vec{q}_1}}{\omega_{\vec{q}_2}^2-(\omega_{\vec{q}_1}+i\omega_n)^2}\right)+\frac{N_{\vec{q}_2}+1}{2\omega_{\vec{q}_2}}\left(\frac{(A_{\vec{q}_1}-\omega_{\vec{q}_2}+i\omega_n)B_{\vec{q}_2}+(A_{\vec{q}_2}-\omega_{\vec{q}_2})B_{\vec{q}_1}}{\omega_{\vec{q}_1}^2-(\omega_{\vec{q}_2}-i\omega_n)^2}\right)$$

$$+\frac{N_{\vec{q}_1}}{2\omega_{\vec{q}_1}}\left(\frac{(A_{\vec{q}_1}+\omega_{\vec{q}_1})B_{\vec{q}_2}+(A_{\vec{q}_2}+\omega_{\vec{q}_1}-i\omega_n)B_{\vec{q}_1}}{\omega_{\vec{q}_2}^2-(\omega_{\vec{q}_1}-i\omega_n)^2}\right)+\frac{N_{\vec{q}_2}}{2\omega_{\vec{q}_2}}\left(\frac{(A_{\vec{q}_2}+\omega_{\vec{q}_2}+i\omega_n)B_{\vec{q}_2}+(A_{\vec{q}_2}+\omega_{\vec{q}_2})B_{\vec{q}_1}}{\omega_{\vec{q}_1}^2-(\omega_{\vec{q}_2}+i\omega_n)^2}\right)\ . \qquad \text{(B2)}$$

*Work supported by the National Science Foundation.

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PHYSICAL REVIEW B VOLUME 3, NUMBER 3 1 FEBRUARY 1971

Canted Spin Phase in Gadolinium Iron Garnet

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(Received 31 August 1970)

The behavior of ferrimagnetic garnets with three magnetic sublattices in an external field has been investigated. In the field-temperature plane we determined the stability limits of the collinear phases with respect to angle formation between the various sublattice moments. The application of our molecular-field analysis to gadolinium iron garnet (GdIG) shows that the angle formation between the two strongly coupled iron sublattices cannot be neglected. By measuring the Faraday effect in fields up to 10 kOe, we were able to observe the angled spin phase in GdIG in the vicinity of the compensation point.

I. INTRODUCTION

A magnetic field applied to a ferrimagnet in which the sublattice moments are collinear tends to align all the moments parallel to itself, in opposition to the exchange interactions which try to maintain the ferrimagnetic antiparallel configuration. Under certain conditions of field and temperature, this competition can result in spin configurations in which the individual sublattice moments form angles with each other and with the field even in isotropic crystals. This phenomenon has been studied in a number of investigations both experimentally and theoretically.[1–4]

In the case of three-sublattice iron garnets, one expects that with increasing field, angles are first formed predominantly between the rare-earth sublattice moment and the net iron moment. Only at much higher fields is the antiparallel alignment of the iron moments broken up. This assumption is based on the fact that the exchange coupling of the rare-earth sublattice to the iron sublattices is weak compared with the dominant coupling between the two iron sublattices. Thus, in the past, molecular-field theory has been applied to calculate the magnetic response of garnets to an external field,[1–3] but angle formation between the two iron moments has been neglected. This simplifies the theoretical treatment, and the instability criterion for the collinear phases is a two-sublattice relationship.[1] Even in small fields, however, the simplified treatment can lead to wrong results, especially if the compensation point does not occur at very low temperatures.

In the present work (Sec. II) we derive the correct instability criterion for a three-sublattice system. This criterion, together with the molecular-field equations, determines the phase boundaries in the field-temperature plane. As an application, we calculate the magnetic-phase diagram for gadolinium iron garnet (GdIG), the magnetic properties of which can be reasonably well described by a simple three-sublattice model. As all magnetic ions in this material have an *S* ground state, crystal-field and anisotropy effects are small[5] and can be neglected.[6] The values of the molecular-field coefficients are available from various sources; in our calculations for GdIG we use Anderson's set[7] which was obtained from a fit of magnetization measurements. The results agree well with our Faraday rotation measurements which are reported in Sec. III. The Faraday rotation is very sensitive to changes in the spin configuration and therefore provides an excellent tool for detection of the boundaries of the